Solution for HW10 1-12-2016

§72) 1) a) $ze^{\frac{1}{z}} = z\sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^{n-1}} = z+1+\frac{1}{2!z}+\frac{1}{3!z^2}+\cdots$

Hence the function $f(z)=ze^{\frac{1}{z}}$ has an essential singularity at $z=0$.

b) $\frac{z^2}{1+z} = \frac{z^2-1+1}{1+z} = \frac{(z+1)(z-1)+1}{1+z} = (z-1)+\frac{1}{1+z}$

Hence the function $f(z)=\frac{z^2}{1+z}$ has a simple pole at $z=-1$.

c) $\frac{\sin z}{z} = \frac{\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)}{z} = 1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots$

Hence the function $f(z)=\frac{\sin z}{z}$ has a removable singularity at $z=0$.

2) a) Note that

$$\frac{1-\cosh z}{z^3} = \frac{-\sum_{n=1}^{\infty}\frac{z^{2n}}{2n!}}{z^3} = -\left(\frac{1}{2!z}+\frac{z}{4!}+\frac{z^3}{6!}+\cdots\right)$$

Hence the function $f(z)=\frac{1-\cosh z}{z^3}$ has a simple pole at $z=0$. Moreover, $\operatorname{Res}_{z=0} f(z) = \frac{-1}{2}$ //.

b) Note that

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left(-(2z)-\frac{(2z)^2}{2!}-\frac{(2z)^3}{3!}-\cdots\right)$$
$$= \frac{-2}{z^3}-\frac{2}{z^2}-\frac{4}{3z}-\cdots$$

Hence the function $f(z)=\frac{1-e^{2z}}{z^4}$ has a pole of order 3 at $z=0$.

Moreover, $\operatorname{Res}_{z=0} f(z) = \frac{-4}{3}$ //.

c) $$\frac{e^{2z}}{(z-1)^2} = \frac{e^2}{(z-1)^2}\left(1+2(z-1)+\frac{[2(z-1)]^2}{2!}+\cdots\right)$$
$$= \frac{e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + 2e^2 + \cdots$$

Hence the function $f(z) = \dfrac{e^{2z}}{(z-1)^2}$ has a pole of order 2 at $z=1$.

Moreover, $\operatorname{Res}_{z=1} f(z) = 2e^2$ //

§14) 1) a) Since at $z=1$, $\phi(z) = z^2+2 = 1^2+2 = 3 \neq 0$, the function $\dfrac{\phi(z)}{z-1}$ has a simple pole at $z=1$ with residue $\phi(1) = 3$ //

b) Since at $z = \frac{-1}{2}$, the function $\phi(z) = z^3 = \frac{-1}{8} \neq 0$, the function $\dfrac{\phi(z)}{8(z+\frac{1}{2})^3}$ has a pole of order 3 at $z = \frac{-1}{2}$ with residue $\dfrac{1}{2!}\left(\dfrac{\phi(z)}{8}\right)^{(2)}\Bigg|_{z=-\frac{1}{2}} = \dfrac{-3}{16}$ //

c) At $z = \pi i$, the function $\phi_{\pi i}(z) = \dfrac{e^z}{z+\pi i} = \dfrac{-1}{2\pi i} \neq 0$.

At $z = -\pi i$, the function $\phi_{-\pi i}(z) = \dfrac{e^z}{z-\pi i} = \dfrac{1}{2\pi i} \neq 0$.

As a result, the function $\dfrac{e^z}{z^2+\pi^2} = \dfrac{e^z}{(z+\pi i)(z-\pi i)}$ has simple poles at $z = \pm\pi i$ with residue $\mp\dfrac{1}{2\pi i}$ respectively.

2) a) $\operatorname*{Res}_{z=-1} \frac{z^{\frac{i}{4}}}{z+1} = (-1)^{\frac{i}{4}} = (e^{i\pi})^{\frac{i}{4}} = e^{i(\frac{\pi}{4})} = \frac{1+i}{\sqrt{2}}$,,

b) $\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \operatorname*{Res}_{z=i} \frac{(\operatorname{Log} z)/(z+i)^2}{(z-i)^2}$

$$= \frac{d}{dz} \frac{\operatorname{Log} z}{(z+i)^2} \bigg|_{z=i}$$

$$= \frac{(z+i)^2 \cdot \frac{1}{z} - \operatorname{Log} z (2)(z+i)}{(z+i)^4} \bigg|_{z=i}$$

$$= \frac{(2i)^2 \frac{1}{i} - (\operatorname{Log} i)(2)(2i)}{(2i)^4}$$

$$= \frac{\pi + 2i}{8} \text{ ,,}$$

3) a) Since the only singularity lying inside $|z-2|=2$ is $z=1$, we have $\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz$

$$= 2\pi i \left(\frac{3z^3+2}{z^2+9} \bigg|_{z=1} \right)$$

$$= 2\pi i \cdot \frac{5}{10}$$

$$= \pi i \text{ ,,}$$

b) Since all the singularities lie inside $|z|=4$, by Cauchy's Residue theorem,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz = 2\pi i \left(\frac{3z^3+2}{z^2+9} \bigg|_{z=1} + \frac{3z^3+2}{(z-1)(z+3i)} \bigg|_{z=3i} + \frac{3z^3+2}{(z-1)(z-3i)} \bigg|_{z=-3i} \right).$$

$$= 2\pi i\left(\frac{1}{2} + \frac{-81i+2}{(3i-1)(6i)} + \frac{81i+2}{(-3i-1)(-6i)}\right)$$

$$= 6\pi i_{\,,,}$$

5) Since all the singularities lie inside $|z|=2$, by Cauchy's Residue theorem,

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}dz = 2\pi i\left(\left.\frac{\cosh \pi z}{z^2+1}\right|_{z=0} + \left.\frac{\cosh \pi z}{z(z+i)}\right|_{z=i}\right.$$
$$\left. + \left.\frac{\cosh \pi z}{z(z-i)}\right|_{z=-i}\right)$$
$$= 2\pi i\left(1 + \frac{(-1)}{i(2i)} + \frac{(-1)}{(-i)(-2i)}\right)$$
$$= 4\pi i_{\,,,}$$

6) c) Note that

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{z^{-3}e^{z}}{1+z^{-3}}\right) = \left(\frac{e^{z}}{1+z^3}\right)\frac{1}{z^2}$$

Since the singularities of $f(z)$ (1, $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$) lie inside the circle $|z|=3$, we have

$$\int_{|z|=3} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right)$$
$$= 2\pi i \left.\frac{d}{dz}\left(\frac{e^z}{1+z^3}\right)\right|_{z=0}$$
$$= 2\pi i \left.\frac{(1+z^3)e^z - e^z(3z^2)}{(1+z^3)^2}\right|_{z=0}$$
$$= 2\pi i_{\,,,}$$

§76) 3) a) Note that $\cos(z_n)=0$ while $\cos'(z_n) = -\sin z_n = (-1)^{n+1}$

Hence $\operatorname*{Res}_{z=z_n} z\sec z = \operatorname*{Res}_{z=z_n} \frac{z}{\cos z} = \frac{z_n}{\cos'(z_n)} = (-1)^{n+1}z_n$,,

b) Note that $\cosh z_n = 0$ while $\cosh'(z_n) = \sinh z_n = 1$.

Hence $\operatorname{Res}_{z=z_n} \tanh z = \operatorname{Res}_{z=z_n} \frac{\sinh z}{\cosh z} = \frac{\sinh z_n}{\cosh z_n} = 1$.

4) a) $$\int_C \tan z \, dz = \int_C \frac{\sin z}{\cos z} dz$$
$$= \left(\operatorname{Res}_{z=\frac{\pi}{2}} \frac{\sin z}{\cos z} + \operatorname{Res}_{z=-\frac{\pi}{2}} \frac{\sin z}{\cos z} \right)(2\pi i)$$
$$= \left(\frac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} + \frac{\sin(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2})} \right)(2\pi i)$$
$$= -4\pi i$$

5) $$\int_{C_N} \frac{dz}{z^2 \sinh z} = 2\pi i \left(\operatorname{Res}_{z=0} \frac{1}{z^2 \sinh z} + \sum_{n=1}^{N} \operatorname{Res}_{z=n\pi} \frac{1}{z^2 \sinh z} + \sum_{n=1}^{N} \operatorname{Res}_{z=-n\pi} \frac{1}{z^2 \sinh z} \right)$$

For $\operatorname{Res}_{z=0} \frac{1}{z^2 \sinh z}$, note that $\frac{1}{z^2 \sinh z} = \frac{1}{z^3 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}$

Let $\phi(z) = \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}$. Note that $\phi(0) = 1 \neq 0$.

$$\phi'(z) = \frac{-\left(-\frac{2z}{3!} + \frac{4z^3}{5!} - \cdots\right)}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2},$$

$$\phi''(z) = \frac{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2 (-1)\left(-\frac{2}{3!} + \frac{4 \cdot 3 z^2}{5!} - \cdots\right)}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^4} - \frac{-\left(-\frac{2z}{3!} + \frac{4z^3}{5!} - \cdots\right)(2)\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)\left(-\frac{2z}{3!} + \cdots\right)}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^4}$$

$$\Rightarrow \phi''(0) = \frac{(1)(-1)(-\frac{2}{3!})}{1^4} = \frac{2}{3!}$$

$$\therefore \operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{\phi''(0)}{2!} = \frac{1}{6}$$

For $\operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z}$, note that $\sin(n\pi) = 0$ while

$\sin'(n\pi) = \cos(n\pi) = (-1)^n \neq 0.$

Hence $\operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(n\pi)^2 (-1)^n} = \frac{(-1)^n}{n^2 \pi^2}$

Note that $\operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z} = \frac{(-1)^n}{n^2 \pi^2} = \operatorname*{Res}_{z=-n\pi} \frac{1}{z^2 \sin z}$.

Hence we have

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right)$$

$$\Rightarrow \lim_{N \to \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = \lim_{N \to \infty} 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right)$$

$$\Rightarrow 0 = \frac{1}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$